

Spectral Differences Between Quantum Evolution and Stochastic Laplacian Dynamics

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Institute of Physics Singapore (IPS) Meeting 2026 | 28–30 October 2026 | Singapore University of Technology and Design (SUTD)

Research Question: If Quantum and Stochastic Laplacian Evolution Systems have formally similar matrix-exponential form, why do they exhibit fundamentally different long-term behaviour?

Quantum Evolution $\psi(t) = e^{-iHt}\psi(0)$	Laplacian Evolution $p(t) = e^{-Lt}p(0)$
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The key structural difference is the extra factor i in quantum evolution. My aim was to understand this difference using explicit calculations on small models.

Effect of Eigenvalue λ

Quantum Evolution: $e^{-i\lambda t}$ Real part $\rightarrow$ oscillation Negative Imaginary part $\rightarrow$ exponential decay	Markov Evolution: $e^{-\lambda t}$ Real part $\rightarrow$ exponential decay Imaginary part $\rightarrow$ oscillation
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In my study, I followed four steps:
-I chose simple model matrices
-Inserted specific numerical parameters
-Calculated their eigenvalues and exact time evolution
-Plotting and numerically checking the resulting behaviour

Non – Hermitian Quantum Model

Consider a two-state non-Hermitian Hamiltonian

$$H = \begin{pmatrix} E - i\gamma & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} 1 - 0.4i & 0 \\ 0 & 1 \end{pmatrix}$$

Example: $E = 1, \gamma = 0.4, \hbar = 1$

Using characteristic eqn.: $\det(\lambda I - H) = 0$
 $\det \begin{pmatrix} \lambda - (1 - 0.4i) & 0 \\ 0 & \lambda - 1 \end{pmatrix} = 0$
Eigenvalues: $\lambda_1 = 1 - 0.4i, \lambda_2 = 1$

$e^{-0.4t}e^{-it} \rightarrow$ negative imaginary part produces exponential decay; real part gives phase oscillation
 $e^{-it} \rightarrow$ pure phase oscillation from real eigenvalue

$$e^{-iHt} = \begin{pmatrix} e^{-i(E-i\gamma)t} & 0 \\ 0 & e^{-iEt} \end{pmatrix}$$
$$e^{-i(E-i\gamma)t} = e^{-\gamma t}e^{-iEt} = e^{-0.4t}e^{-it}$$
$$e^{-iEt} = e^{-it}$$

Using: $\psi(t) = e^{-iHt}\psi(0)$
and initial state : $\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\psi(t) = \begin{pmatrix} e^{-0.4t}e^{-it} & 0 \\ 0 & e^{-it} \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\psi(t) = \frac{e^{-it}}{\sqrt{2}} \begin{pmatrix} e^{-0.4t} \\ 1 \end{pmatrix}$$

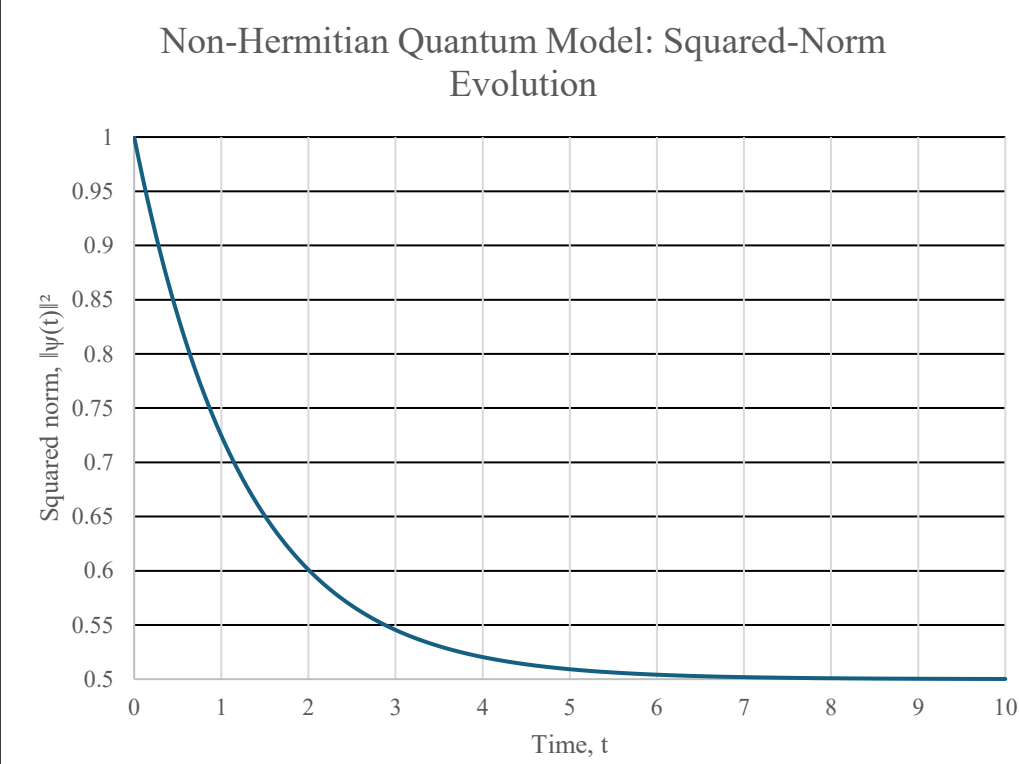


Fig1: Squared-norm decay from 1 toward 0.5

At t=0, squared norm is 1
At t=5, squared norm is ~ 0.509 and drops to 0.5 as $t \rightarrow \infty$

With a complex eigenvalue, the non-Hermitian model shows decay with phase oscillation. The negative imaginary part causes decay, while the real part causes phase evolution.

Squared norm: $||\psi(t)||^2 = \frac{1+e^{-0.8t}}{2}$, $||\psi(t)||^2 \rightarrow 0.5$ as $t \rightarrow \infty$

Two-State Continuous-Time Markov Model

Consider the generator (Laplacian)

$$L_2 = \begin{pmatrix} \gamma & -\delta \\ -\gamma & \delta \end{pmatrix} = \begin{pmatrix} 0.3 & -0.1 \\ -0.3 & 0.1 \end{pmatrix}$$

Example: $\gamma = 0.3$ ($1 \rightarrow 2$), $\delta = 0.1$ ($2 \rightarrow 1$)

using characteristic eqn., $\det(\lambda I - L_2) = 0$
 $\det \begin{pmatrix} \lambda - 0.3 & 0.1 \\ 0.3 & \lambda - 0.1 \end{pmatrix} = 0$
Eigenvalues: $\lambda_1 = 0, \lambda_2 = 0.4$

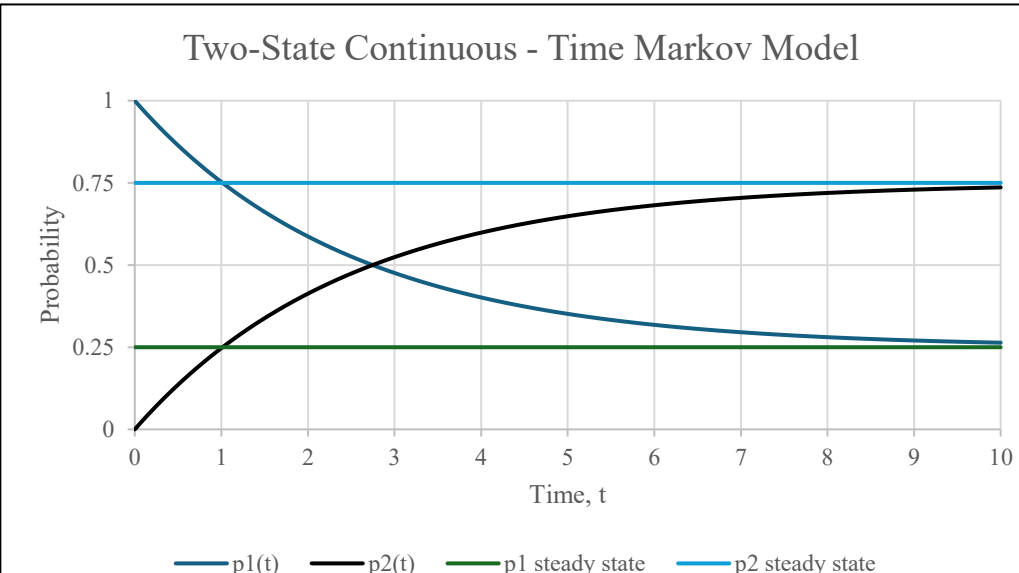


Fig2: Non-oscillatory relaxation toward the stationary state

$e^{-0t} = \mathbf{1} \rightarrow$ stationary state
 $e^{-0.4t} \rightarrow$ no imaginary part, decaying, non-oscillation mode

My model transition rates are: $\gamma = 0.3, \delta = 0.1$
Calculating stationary probabilities: $p_1 = \frac{0.1}{0.3+0.1}$ and $p_2 = \frac{0.3}{0.3+0.1}$
i.e. $p_1 = 0.25, p_2 = 0.75$
Using $\frac{dp}{dt} = -L_2p, \frac{d}{dt} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, p_\infty = \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$
Using initial state: $p_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
Deriving time-dependent eqn.: $p_1(t) - 0.25 = Ce^{-0.4t}$ where $p_1(0) = 1, C = 0.75$
 $p_1(t) = 0.25 + 0.75e^{-0.4t}$
Since, $p_1(t) + p_2(t) = 1$
 $p_2(t) = 0.75(1 - e^{-0.4t})$
With only real eigenvalues, the two-state transient decays without oscillation
The zero eigenvalue gives a stationary state, while the positive eigenvalue produces exponential decay without oscillation.

References
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[2] Merris, R. (1995). *A survey of graph Laplacians*. Linear and Multilinear Algebra, 39(1–2), 19–31.
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Acknowledgement: I gratefully acknowledge Prof. Ching Hua Lee for his mentorship, guidance and valuable feedback on the development of this work.

Three-State Continuous-Time Non-Reversible Markov Model

My three-state model has no reverse transitions. State $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$

Probability vector, $p(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix}$

$\frac{dp(t)}{dt} = -L_3p(t)$

Generator (Laplacian):

$$L_3 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Each column of L_3 sums to zero.
Chain is non-reversible, one-way cycle violates detailed balance.

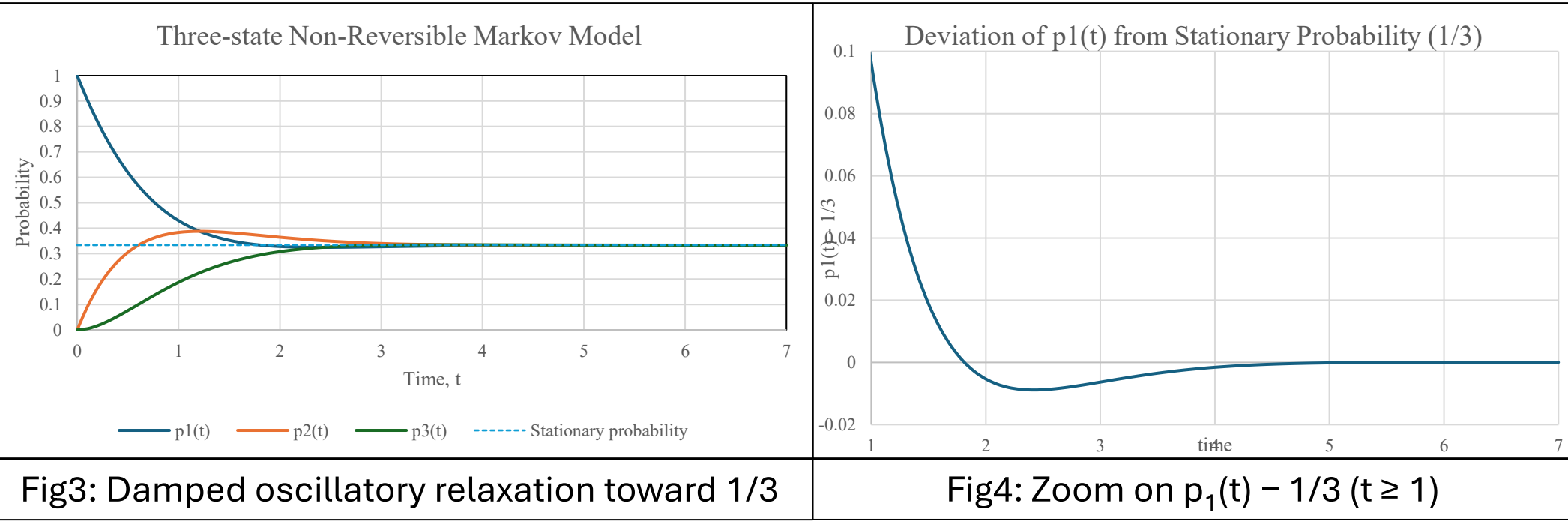
Using characteristic eqn.: $\det(\lambda I - L_3) = 0$
 $\det \begin{pmatrix} \lambda - 1 & 0 & 1 \\ 1 & \lambda - 1 & 0 \\ 0 & 1 & \lambda - 1 \end{pmatrix} = 0$
Eigenvalues: $\lambda_0 = 0, \lambda_{\pm} = \frac{3}{2} \pm i\frac{\sqrt{3}}{2}$

$e^{-0t} = \mathbf{1} \rightarrow$ stationary state.
1st and 2nd part together, the complex eigenvalues gives damped oscillation.
1st part, $e^{-\frac{3}{2}t} \rightarrow$ exponential decay.
2nd part, $e^{\mp\frac{\sqrt{3}}{2}it} \rightarrow$ oscillation.

With a complex-conjugate eigenvalue pair, the three-state transient shows damped oscillations. The positive real part produces decay, while the imaginary part produces oscillation

Stationary state: $L_3p_\infty = 0$, gives $p_1 = p_2 = p_3; p_1 + p_2 + p_3 = 1$. Hence, $p_\infty = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$

All probabilities moving toward 1/3 with small overshoots and undershoots.



Zooming into the oscillatory component

Using initial state: $p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and the complex eigenvalues $\lambda_{\pm} = \frac{3}{2} \pm i\frac{\sqrt{3}}{2}$
The complex modes give a real solution containing sine and cosine. Hence,
 $\lambda_{\pm} \rightarrow e^{-3t/2} \left[A \cos\left(\frac{\sqrt{3}}{2}t\right) + B \sin\left(\frac{\sqrt{3}}{2}t\right) \right]$; using $A = \frac{2}{3}, B = 0$
$$p_1(t) = \frac{1}{3} + \frac{2}{3}e^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{3}}{2}t\right)$$

Stationary value, i.e. $1/3 +$ temporary deviation, where $e^{-\frac{3t}{2}}$ makes temporary part get smaller, causing damping and ultimately only $1/3$ remains. The sine and cosine causes up and down movement.

At the beginning $(p_1, p_2, p_3) = (1, 0, 0)^T$, with probability flowing $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$; p_1 falls, p_2 rises, p_3 rises slightly later (Fig3).
 p_1 crosses $1/3$ at $t = \pi/\sqrt{3} \approx 1.81$ and reaches a minimum of 0.32 ($< 1/3$) at $t \approx 2.42$, then comes back (Fig4).

Crossing the stationary value is the oscillation; the factor $e^{-\frac{3t}{2}}$ makes it die out, so all three probabilities settle at $1/3$.

Conclusion

The dynamics are determined by both the spectrum and the form of the evolution operator in which that spectrum appears.

$$e^{-i\lambda t} \text{ vs. } e^{-\lambda t}$$

The additional factor i changes the dynamical roles of the real and imaginary parts of the spectrum.

Non- Hermitian quantum model: a negative imaginary component of the eigenvalue produces exponential decay.
Two-state Markov model: a real spectrum produces non-oscillatory relaxation.
Three-state Markov model: a complex-conjugate pair with positive real part produces damped oscillations.

The computed examples show directly how real and imaginary spectral parts contribute differently to the time evolution of the quantum and stochastic models.

Scope

This study uses small finite-dimensional models with explicit numerical parameters. My next step is to investigate how these spectral relationships extend to larger quantum and stochastic systems using numerical computation.

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